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LETTER TO THE EDITOR

Graph partitioning and dilute spin glasses: the minimum cost solution

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Abstract. We calculate the ground-state cost function/energy for graph bipartitioning and spin glasses on networks with fixed finite valences, allowing a continuous distribution for the effective field. Compared with previous calculations using a field distribution in integral multiples of the coupling strength J, the ground-state energy is higher by less than one per cent, although the distribution itself is altered drastically.

The graph bipartitioning problem [1] is a typical example of applying techniques from the theory of spin glasses to study complex optimisation. In this problem, one has a set of randomly connected vertices (which we call a graph) and the issue is to partition them into two subsets of equal size, so that the number of connections between the two sets (or cost function) is minimised. This is equivalent to finding the ground state of a randomly connected ferromagnetic Ising model subject to the constraint that the total magnetisation is zero [2]. It is also related to the problem of a spin glass on a random network.

Our concern is with averages over ensembles of statistically equivalent graphs, with particular regard to the average minimal cost or ground-state energy, and with the limit as the total number of vertices N tends to infinity. Such solutions, analytic or simulational, have been proposed for several different types of graph [1, 3–7]. In this letter, we concentrate on the case in which each vertex of the graph is connected to exactly c other vertices and c is independent of N (we refer to this as fixed intensive valence).

In an earlier letter [7] it was shown that, within replica-symmetric theory, the cost function is determined by the distribution of an auxiliary field, which obeys a self-consistency equation equivalent to that for the effective field due to descendents of Ising spins on a corresponding Bethe lattice. In that study, as in related studies for graphs of average intensive valence [4, 8], the self-consistent field equation was solved subject to the assumption that the auxiliary field at the ground state could only take values which are integral multiples of the coupling strength J. This yielded values for the minimal cost a few per cent lower than those obtained by simulation [5]. In the Bethe lattice there exists, however, another solution with both a continuous part and delta-function peaks at integral multiples of J [9–11]. Indeed, it has recently been shown in the case of an averaged finite-valenced network [12] that the solution with delta-function peaks only is unstable against the introduction of a continuous field distribution.

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In this letter we evaluate the stable field distribution for a variety of values of c and investigate the consequences for the minimal cost. We find that the distribution is changed significantly compared with the integral field approximation, but the ground-state energy is increased by less than one per cent. We also consider further the relationship with the Bethe lattice, showing that the Bethe approximation for the free energy is exact for this graph partitioning problem.

We start by re-iterating the relation of the graph bipartitioning problem to its magnetic analogue and highlighting the relevant fundamental formulae. With each vertex i we associate an Ising spin S_i which takes the value $+1$ if the vertex belongs to one set of the bipartitioning and -1 if it belongs to the other. The number of connections N_{ct} between the two sets, which is the cost function to be minimised, is then related to the energy E of the corresponding ferromagnet,

$$E = -J \sum_{(ij)} S_i S_j \tag{1}$$

where (ij) denotes a pair of connected vertices, via the relation

$$N_{ct} = E/2J + \tfrac{1}{4}cN. \tag{2}$$

Since the two sets have to be of equal size, the total magnetisation is constrained to be zero. Henceforth we shall employ the language of the magnetic analogue.

Within the replica-symmetric ansatz, the spin configuration at a temperature $T = \beta^{-1}$ is described by the local field distribution

$$P(h) = \prod_{i=1}^{c} \left(\int d\Phi_i \pi(\Phi_i) \right) \delta\left(h - \sum_{i=1}^{c} \xi(\Phi_i) \right) \tag{3}$$

where $\pi(\Phi)$ is an auxiliary field distribution satisfying the self-consistency equation

$$\pi(\Phi) = \prod_{i=1}^{c-1} \left(\int d\Phi_i \pi(\Phi_i) \right) \delta\left(\Phi - \sum_{i=1}^{c-1} \xi(\Phi_i) \right) \tag{4}$$

and $\xi(\Phi)$ is the function

$$\xi(\Phi) = \beta^{-1} \tanh^{-1}(\tanh \beta J \tanh \beta\Phi). \tag{5}$$

Both $P(h)$ and $\pi(\Phi)$ are even functions of their arguments. The free energy per site f is completely determined by $\pi(\Phi)$,

$$\begin{aligned} -\beta f = -\tfrac{1}{2}c \ln \cosh \beta J + c \int d\Phi P(\Phi) \ln(1 - \tanh^2 \beta J \tanh^2 \beta\Phi) \\ - c \int d\Phi_1\, d\Phi_2\, \pi(\Phi_1)\pi(\Phi_2) \ln(1 + \tanh \beta J \tanh \beta\Phi_1 \tanh \beta\Phi_2) \\ + \int dh\, P(h) \ln 2 \cosh \beta h. \end{aligned} \tag{6}$$

The ground-state energy follows from the limit of the free energy as $T \to 0$.

Before discussing the solution to (4) let us comment on the relation of the above results to a system on a Bethe lattice. Earlier [7], we pointed out that (4) is identical with that for the distribution of fields due to descendents on a Bethe lattice of Ising spins with ferromagnetic exchange J [13], with the constraint that $\pi(\Phi)$ is even, reflecting a zero magnetisation constraint; explicitly, the $\xi(\Phi_i)$; $i = 1, \ldots, c$ correspond to the individual fields due to first descendents at a site on the Bethe lattice. $P(h)$ is the total field distribution. These field distributions are also those appropriate to a $\pm J$ Ising spin glass on a Bethe lattice [13, 14], where the exchange disorder ensures that the even solution is the relevant one.

Let us now consider the relation of f to a Bethe solution. If we consider a site on the Bethe lattice and represent the effects of the rest of the lattice by a local field h, then the average free energy per site is given by

$$-\beta f_{\text{site}} = \int \mathrm{d}h\, P(h) \ln 2 \cosh \beta h. \tag{7}$$

On the other hand, if we consider a bond on the Bethe lattice while representing the effects of the rest of the lattice by the effective fields Φ_1 and Φ_2 on the bond-ended sites, the average free energy per bond is given by

$$\begin{aligned} -\beta f_{\text{exch}} = \int & \mathrm{d}\Phi_1\, \mathrm{d}\Phi_2\, \pi(\Phi_1)\pi(\Phi_2) \\ & \times \ln[2\, \mathrm{e}^{\beta J} \cosh \beta(\Phi_1+\Phi_2) + 2\mathrm{e}^{-\beta J} \cosh \beta(\Phi_1-\Phi_2)]. \end{aligned} \tag{8}$$

The Bethe approximation to the free energy of a real lattice with b bonds per site is then given by

$$f = bf_{\text{exch}} - (2b-1) f_{\text{site}}. \tag{9}$$

For the fixed valence network under consideration $b = \frac{1}{2}c$ and equations (6) and (9) are identical. Thus for this problem the Bethe approximation is exact, at least at the replica-symmetric level.

We now proceed to evaluate explicitly the $T=0$ field distribution and the ground-state energy of the network. At $T=0$ the function $\xi(\Phi)$, equation (5), takes an especially simple form:

$$\xi(\Phi) = \begin{cases} J & \Phi \geqslant J \\ \Phi & -J < \Phi < J \\ -J & \Phi \leqslant -J \end{cases} \tag{10}$$

so that (4) has a solution in which Φ only takes values which are integral multiples of J. Since this solution is unstable against continuous fluctuations [9–12] we now consider the more general stable solution.

It is more convenient to consider the field distribution $\mathbb{P}(\xi)$ which is related to $\pi(\Phi)$ by

$$\mathbb{P}(\xi)\, \mathrm{d}\xi = \pi(\Phi)\, \mathrm{d}\Phi \tag{11}$$

where ξ and Φ are related by (5). The self-consistency equation for $\mathbb{P}(\xi)$ is therefore

$$\mathbb{P}(\xi) = \prod_{i=1}^{c-1} \left(\int \mathrm{d}\xi_i \mathbb{P}(\xi_i) \right) \delta\left(\xi - \beta^{-1} \tanh^{-1}\left(\tanh \beta J \tanh \beta \sum_{i=1}^{c-1} \xi_i \right)\right) \tag{12}$$

In this letter, we propose a sequence of successive approximations to the true ground state which converges very rapidly. The key observation is that, by virtue of equation (10), a distribution function $\mathbb{P}(\xi)$, with ξ being integral multiples of J/M (M integral) only, is a possible self-consistent solution to (12). In fact, our previous result corresponds to the case $M=1$. In the limit $M \to \infty$, we expect the approximation to approach the continuous distribution of the true ground state. As we shall see, the ground-state energy acquires four-figure precision for M less than 100.

To be specific, let us write the distribution function $\mathbb{P}(\xi)$ as

$$\mathbb{P}(\xi)=\pi_0\delta(\xi)+\sum_{i=1}^{M}\pi_i\left[\delta\left(\xi-\frac{i}{M}J\right)+\delta\left(\xi+\frac{i}{M}J\right)\right] \tag{13}$$

where the π_i satisfy the constraint

$$\pi_0+2\sum_{i=1}^{M}\pi_i=1. \tag{14}$$

The self-consistency equation (9) then becomes

$$\begin{aligned}\pi_i&=\sum_{j_1+\ldots+j_k=i}(\pi_{j1}\ldots\pi_{jk}) \qquad 0\leq i<M\\ \pi_M&=\sum_{j_1+\ldots+j_k\geq M}(\pi_{j1}\ldots\pi_{jk})=\tfrac{1}{2}(1-\pi_0)-\sum_{i=1}^{M-1}\pi_i\end{aligned} \tag{15}$$

where $k=c-1$, and each of the variables $j_1,\ldots,j_k$ runs from $-M$ to M with the convention $\pi_{-j}=\pi_j$. Similarly, the local field distribution is given by

$$P(h)=p_0\delta(h)+\sum_{i=1}^{cM}p_i\left[\delta\left(h-\frac{i}{M}J\right)+\delta\left(h+\frac{i}{M}J\right)\right] \tag{16a}$$

where

$$p_i=\sum_{j_1+\ldots+j_c=i}(\pi_{j_1}\ldots\pi_{j_c}). \tag{16b}$$

From (6), the ground-state energy is then given by

$$-\frac{E}{NJ}=\frac{c}{2}-\frac{2c}{M}\sum_{i=1}^{M}i\pi_i+\frac{2c}{M}\sum_{i=1}^{M}\sum_{j=1}^{M}\pi_i\pi_j\min(i,j)+\frac{2}{M}\sum_{i=1}^{cM}ip_i. \tag{17}$$

Figures 1(a)-(d) show the resulting field distribution $\{\pi_i\}$ for a trivalent network for a variety of choices of M. It can be seen that, as M increases, the peaks at $\xi=0$ and

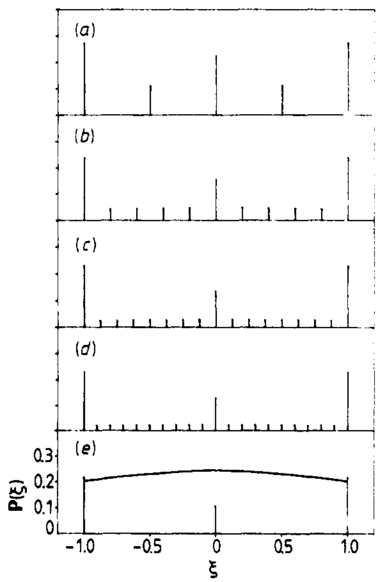

Figure 1. The field distribution $\{\pi_i\}$ for a trivalent network. (a) $M=2$, (b) $M=5$, (c) $M=8$, (d) $M=10$, (e) the function (15) in the $M\to\infty$ limit.

$\pm J$ become more and more pronounced, suggesting a delta-function component in the continuous field limit. The heights of the other peaks are relatively constant.

It is therefore expedient to write $\mathbb{P}'(\xi)$ in the continuous limit as

$$\mathbb{P}'(\xi)=f_0\delta(\xi)+f_1(\delta(\xi-J)+\delta(\xi+J))+f(\xi) \tag{18}$$

where $f(\xi)$ is a continuous function. Table 1 shows the numerical results for f_0, f_1 and $f(\xi)$ at representative points for a trivalent network. They are related to the $\{\pi_i\}$ via the relations

$$f_0=\lim_{M\to\infty}(\pi_0-\pi_1) \tag{19a}$$

$$f_1=\lim_{M\to\infty}(\pi_M-\tfrac{1}{2}\pi_{M-1}) \tag{19b}$$

$$\lim_{M\to\infty} f(i/M)=\pi_i M. \tag{19c}$$

The function $\mathbb{P}'(\xi)$ in the continuous limit is plotted in figure 1(e). Table 1 also shows the numerical result for the ground-state energy using (17). It is evident that the data approach the continuous limit rapidly. In fact, it acquires a four-figure precision for M less than 100. We have checked these results by assuming a continuous $\mathbb{P}(\xi)$ and computing the integral in equation (4) numerically at each step of the iteration. The results are in good agreement.

Numerical results for networks of other valences are shown in table 2. It should be noticed that networks with even valences do not have the delta-function peak at the zero field position.

It is observed that the fraction of zero field sites, as indicated by f_0 for odd valences, is greatly reduced when compared with the integral field case. Take, for example, the

Table 1. Values of f_0, f_1, $f(\xi)$ and the ground-state energy for a trivalent network.

M	f_0	$f(0.2)$	$f(0.4)$	$f(0.6)$	$f(0.8)$	f_1	$-E/NJ$
5	0.1076	0.2403	0.2338	0.2251	0.2144	0.2180	1.2750
10	0.1070	0.2408	0.2343	0.2256	0.2148	0.2183	1.2749
20	0.1069	0.2410	0.2345	0.2257	0.2149	0.2184	1.2749
40	0.1068	0.2410	0.2345	0.2257	0.2149	0.2144	1.2749
100	0.1068	0.2410	0.2345	0.2257	0.2149	0.2184	1.2749

Table 2. Field distribution and ground-state energy for networks of valences $3\leqslant c\leqslant 9$. E_{cont} and E_{int} are the ground-state energies in the continuous and integral field ansatz, respectively.

c	f_0	f_1	$\frac{1}{2}\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi$	$-E_{\text{cont}}/NJ$	$-E_{\text{int}}/NJ$
3	0.1068	0.2184	0.2282	1.2749	1.2778
4		0.2960	0.2040	1.4833	1.4880
5	0.0462	0.2926	0.1843	1.6849	1.6911
6		0.3301	0.1699	1.8478	1.8554
7	0.0261	0.3290	0.1580	2.0129	2.0290
8		0.3516	0.1484	2.1528	2.1613
9	0.0166	0.3515	0.1402	2.2955	2.3058

trivalent network. In the integral field approximation a site has zero field so long as its two descendents have opposing fields. In the case of a continuous field, however, a site does not necessarily have a zero field even if its two descendents have opposing fields; if the field on the descendent sites are, say, $+J$ and $-0.5J$, the field on the site is $+0.5J$ and is still polarised. We therefore expect the fraction of sites with zero field to be greatly reduced.

As the valence c increases, the weight of the continuous distribution decreases. Furthermore, the peak f_0 approaches zero as the valence c increases. We note in passing that when the valence is extensive, no crazy spins are present. The peak f_1, representing all the contributions for $J < \Phi \leq (c-1)J$, increases with c. It is also interesting to note that the value of f_1 for valences $2n$ and $2n+1$ are very close.

Finally we compare the ground-state energy with our previous calculation using the integral field ansatz [7]. It is found that the continuous field distribution, though drastically different from the integral field distribution, nevertheless gives ground-state energies nearly the same as the latter. In fact, the ground-state energy $E_{\rm cont}$ is higher than $E_{\rm int}$ by less than 1 per cent. We conclude that the integral field approximation, though unstable, nevertheless gives very good ground-state energy estimates.

We conclude this report by the following remarks. First, the ground-state solution to the Viana–Bray model of a spin glass [4, 8, 16] and the corresponding problem of equipartitioning graphs of average intensive valence should also be modified in the same way. Thus the assumption that the effective field consists of peaks at integral multiples of J should likewise be replaced by one consisting of both delta-function peaks and a continuous distribution [12]. The second remark concerns the replica-symmetry-breaking solution of the finite-valenced network. Recent results for the spin glass on the Bethe glass indicate that replica symmetry is broken at zero temperature [17, 18], and our solution in this letter is apparently only a first step in finding the true ground state of the system. It is therefore desirable to examine how the effective field distribution and the free energy are modified accordingly. Such a study will undoubtedly be a topic of interest in the near future.

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Note added in proof. Since this letter went to press we have received a preprint by Katsura *et al* [19] which, in the context of the Bethe approximation to a $\pm J$ Ising spin glass, also gave solutions for $\mathbb{P}(\xi)$ for $c=3$, $M=1, 2, 3, 4$.

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